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A q-deformed Poisson distribution based on orthogonal polynomials

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Abstract. We introduce certain q-deformations of the binomial distribution and the Poisson distribution by virtue of q-deformed sequences of orthogonal polynomials. They correspond to the case of free independence for $q = 0$, and are reduced to the usual commutative case when $q = 1$. Furthermore, we determine the probability measure for a q-deformed Poisson distribution by using the formulae for the Al-Salam–Chihara polynomials of Askey and Ismail.

1. Introduction

A non-commutative or quantum probability space is a unital (possibly non-commutative) algebra, $\mathcal{A}$ together with a linear functional, $\phi : \mathcal{A} \to \mathbb{C}$, such that $\phi(1) = 1$. If $\mathcal{A}$ is a C^*-algebra and ϕ is a state then we call a non-commutative probability space $(\mathcal{A}, \phi)$ a C^*-probability space. $\mathcal{A}$ corresponds to the algebra of measurable functions and hence an element in $\mathcal{A}$ is regarded as a non-commutative random variable. The distribution of $x \in \mathcal{A}$ under ϕ is determined as the linear functional on $\mathbb{C}[X]$ (the polynomials in one variable) by

$$\mathbb{C}[X] \ni f \longmapsto \phi(f(X)) \in \mathbb{C}. \tag{1.1}$$

Considered in the C^*-probability context, the distribution of a self-adjoint element in $\mathcal{A}$ can be realized as the probability measure on $\mathbb{R}$.

In recent years the following question has been considered in many papers: what distribution will be obtained in a non-commutative central limit, that is in the case where we replace the classical commutative notion of independence by some other type in a non-commutative probability space. For free independence which was introduced by Voiculescu in [1], the Gaussian distribution is replaced by Wigner's semicircle distribution, which is called the free central limit theorem (see, for instance, [2]). Bożejko, Kümmerer and Speicher introduced q-analogues of Brownian motion and Gaussian processes in [3–5], which are governed by classical independence for $q = 1$ and free independence for $q = 0$. Van Leeuwen and Maassen also investigated a q-deformed Gaussian distribution in [6], which takes the semicircle distribution for $q = 0$ and recovers the Gaussian distribution for $q = 1$. Their constructions were based on $\mathcal{F}_q(\mathcal{H})$, the q-deformation of the Fock space over a Hibert space $\mathcal{H}$. They regarded the distribution of the operator $a(\xi) + a(\xi)^*$ under the vacuum vector state ϕ as the q-deformed Gaussian distribution in a non-commutative probability space $(\Gamma_q(\mathcal{H}), \phi)$, where $a(\xi)$ and $a(\xi)^*$ are the annihilation and the creation operators associated with $\xi \in \mathcal{H}$ satisfying the q-commutation relation, respectively. Furthermore, it is very worthwhile noting

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that this q-deformed Gaussian distribution can be associated with the q-Hermite polynomials. By virtue of this, one can see that, for $\|\xi\| = 1$, the q-deformed Gaussian distribution is supported on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ with the density

$$f(t) = \frac{1}{\pi}\sqrt{1-q}\,\sin\theta \prod_{n=1}^{\infty}(1-q^n)\left|1-\mathrm{e}^{\mathrm{i}2\theta}q^n\right|^2 \tag{1.2}$$

where $\theta \in [0, \pi]$ is such that $t = (2/\sqrt{1-q})\cos\theta$ (for more details see [3]). We should mention here that Nica has found [7] a nice q-analogue of the cumulants generating function $R_q(z)$ which takes Voiculescu's R-transform for the free convolution in the limit $q \to 0$ and recovers the classical cumulants generating function, the logarithm of the Fourier transform, if one takes the limit $q \to 1$. He has also investigated the q-deformed convolution in terms of $R_q(z)$ and the central limit theorem, in which the q-deformed Gaussian distribution appears as its limit distribution.

In the literature, a certain q-deformed Poisson distribution is known as the Euler distribution or Heine distribution as in [8] which, of course, are natural deformations in some sense and all of these, however, are discrete. Kemp [9] showed that the Euler and Heine distributions are the limiting forms of some kind of a q-analogue of the negative binomial distribution and one of the binomial distribution, respectively. Another kind of q-analogue of the binomial distribution has been introduced by Sicong [10] which takes the Euler distribution as its limiting form.

It is natural to regard the distribution of a sum of a free family of projections as the free analogue of the binomial distribution because a projection corresponds to the Bernoulli distribution. In [11], Akiyama and the second author have studied its combinatorial structure and derived the sequence of orthogonal polynomials associated with the free analogue of the binomial distribution, of which a three-term recurrence relation is of the constant coefficients type of Cohen–Trenholme [12]. They have also shown the free Poisson limit and the free de Moivre–Laplace limit by using the recurrence relation of the orthogonal polynomials for the free analogue of the binomial distribution.

Inspired by the above, we would like to introduce different q-deformed binomial and Poisson distributions based on orthogonal polynomials in this paper. We first introduce a q-deformed binomial distributions by virtue of a q-deformed sequence of orthogonal polynomials, which takes the free binomial distribution in the limit $q \to 0$ and reduces to the usual binomial distribution when $q \to 1$. Furthermore, we see that it is compatible with the q-deformed Gaussian distribution if we take the de Moivre–Laplace limiting procedure. By taking the Poisson limit in our q-deformed binomial distribution, we obtain our q-deformed Poisson distribution which is not discrete but has an absolute continuous part, and also find its probability measure by using the formulae for the Al-Salam–Chihara polynomials of Askey and Ismail [13]. We also discuss the fact that the limiting forms of our q-deformed Poisson distribution for $q \to 0$ and for $q \to 1$ can be the free and the usual Poisson distributions, respectively.

2. A q-deformed binomial distribution

Throughout this paper, we make use of the terms of q-calculus, which is over a century old. The standard references on q-calculus are [14, 15]. Let us just remind the reader of some of its basic notation.

We put for $n \in \mathbb{N}_0$ and $q \in [0, 1)$

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1} \qquad \text{with} \quad [0]_q := 0 \tag{2.1}$$

and call it a q-number. Then we have the q-factorial

$$[n]_q! := [1]_q[2]_q \cdots [n]_q \qquad \text{with} \quad [0]_q! := 1. \tag{2.2}$$

Another symbol used is the q-analogue of the Pochhammer symbol,

$$(a;q)_n := \prod_{j=0}^{n-1}(1-aq^j) \qquad \text{in particular} \quad (a;q)_\infty := \prod_{j=0}^{\infty}(1-aq^j) \tag{2.3}$$

where we use the convention that $(a;q)_0 := 1$. A product $(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n$ is denoted as $(a_1,a_2,\ldots,a_r;q)_n$.

There is a q-deformation of the exponential function defined as

$$\exp_q(x) := \sum_{n=0}^{\infty}\frac{x^n}{[n]_q!} \tag{2.4}$$

which satisfies the relation

$$\prod_{n=0}^{\infty}(1-(1-q)q^n x)^{-1} = \exp_q(x). \tag{2.5}$$

Now we shall recall the basic facts on orthogonal polynomials. Let ν be a probability measure on $\mathbb{R}$ with finite moments of all orders. Then it is well known [16] that there exists two sequences of real numbers $\alpha_m \in \mathbb{R}$ and $\beta_m \geqslant 0$, which shall be called the Jacobi parameters, such that the sequence of the orthogonal polynomials $\{P_m(X)\}$ with respect to the measure ν can be given by the recurrence relation,

$$\begin{aligned} &P_0(X) = 1 \qquad P_1(X) = X - \alpha_0 \\ &P_{m+1}(X) = (X-\alpha_m)P_m(X) - \beta_m P_{m-1}(X) \qquad (m \geqslant 1). \end{aligned} \tag{2.6}$$

Moreover, they satisfy that

$$\int_{t\in\mathbb{R}} P_k(t)P_m(t)\,\mathrm{d}\nu(t) = \delta_{k,m}\beta_1\beta_2\cdots\beta_m. \tag{2.7}$$

The Jacobi parameters are determined only by the sequence of moments of ν. Conversely, given the parameters α_m and β_m, Favard's theorem ensures the existence of the probability measure for which the sequence of polynomials determined by the above recurrence relation are orthogonal. It also can be shown that the probability measure ν is supported only in finitely many points if and only if $\beta_m = 0$ from some m on, thus the sequence of polynomials is essentially finite.

Let $\nu_{(n,p)}$ be the probability measure for the binomial distribution $B(n,p)$,

$$\nu_{(n,p)}(\mathrm{d}t) = \sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}\delta_k \tag{2.8}$$

where $\mathrm{d}t$ is the Lebesgue measure and δ_k denotes the Dirac unit mass at $t=k$. The orthogonal polynomials for $\nu_{(n,p)}$ are well known as classical discrete orthogonal polynomials, namely the Krawtchouk polynomials (see, for example, [17, 18]), determined by the following recurrence relation:

$$\begin{aligned} &P_0(X) = 1 \qquad P_1(X) = X - np \\ &P_{m+1}(X) = \left(X-\alpha_m^{(n,p)}\right)P_m(X) - \beta_m^{(n,p)}P_{m-1}(X) \end{aligned} \tag{2.9}$$

with the Jacobi parameters

$$\alpha_m^{(n,p)} = np + (1-2p)m \qquad \beta_m^{(n,p)} = m(n-m+1)p(1-p) \tag{2.10}$$

for $m = 1, 2, \ldots, n$, where $\beta_m^{(n,p)} = 0$ for $m \geqslant n+1$.

Before making a deformation on the orthogonal polynomial for the binomial distribution, we would like to recall the recurrence relation of the orthogonal polynomial for the free binomial distribution $B_{free}(n, p)$, of which a three-term recurrence relation is of the constant coefficients type of Cohen–Trenholme [12]. In [11], Akiyama and the second author have studied the combinatorial structure of the operator,

$$x = \kappa(p_1 + p_2 + \cdots + p_n) \tag{2.11}$$

for a free family of projections $\{p_i\}_{i=1}^n$ $(n \geqslant 2)$ with $\phi(p_i) = \alpha$ $(i = 1, 2, \ldots, n)$ and scalar κ in a C^*-probability space $(\mathcal{A}, \phi)$ and they have obtained the three-term recurrence relation for the distribution of the operator x, namely

$$\begin{aligned} &P_0(X) = \frac{n}{n-1} \qquad P_1(X) = X - \kappa n\alpha \\ &P_{m+1}(X) = (X - \kappa n\alpha - \kappa(1-2\alpha))\, P_m(X) \\ &\qquad\qquad -(n-1)\kappa^2\alpha(1-\alpha)P_{m-1}(X) \qquad (m \geqslant 1) \end{aligned} \tag{2.12}$$

where we should note that the normalization of P_0 is different from that in (2.6). In the above formula, by putting $\kappa = 1$ and rewriting $\alpha = p$ we have the recurrence relation of the orthogonal polynomials for $B_{free}(n, p)$,

$$\begin{aligned} &P_0(X) = 1 \qquad P_1(X) = X - np \\ &P_2(X) = (X - np - (1-2p))\, P_1(X) - np(1-p)P_0(X) \\ &P_{m+1}(X) = (X - np - (1-2p))\, P_m(X) \\ &\qquad\qquad -(n-1)p(1-p)P_{m-1}(X) \qquad (m \geqslant 2). \end{aligned} \tag{2.13}$$

Having this formula in mind, we shall make a deformation on the Jacobi parameters for the binomial distribution in (2.10) and thus define a q-deformed binomial distribution.

Definition 2.1. *For $q \in [0, 1)$, we call the probability measure $(\nu_{(n,p)})_q$ induced from the Jacobi parameters*

$$\begin{aligned} &\left(\alpha_m^{(n,p)}\right)_q = np + (1-2p)[m]_q \\ &\left(\beta_m^{(n,p)}\right)_q = [m]_q\left(n - [m-1]_q\right) p(1-p) \end{aligned} \tag{2.14}$$

the q-deformed binomial distribution and denote $B_q(n, p)$. Here we shall put $\left(\beta_m^{(n,p)}\right)_q = 0$ if $[m-1]_q \geqslant n$.

Example 2.1. In the limit $q \to 0$, it holds that

$$\begin{aligned} &\lim_{q\to 0}\left(\alpha_m^{(n,p)}\right)_q = np + (1-2p) \qquad \text{for} \quad m \geqslant 1 \\ &\lim_{q\to 0}\left(\beta_m^{(n,p)}\right)_q = \begin{cases} (n-1)p(1-p) & \text{if} \quad m \geqslant 2 \\ np(1-p) & \text{if} \quad m = 1. \end{cases} \end{aligned} \tag{2.15}$$

Hence this q-deformed binomial distribution becomes the free binomial distribution when $q = 0$. The probability measure $(\nu_{(n,p)})_0$ can be given as follows (see section 2 in [11]):

$$(\nu_{(n,p)})_0 = \frac{-n\sqrt{-(t-\gamma_-)(t-\gamma_+)}}{2\pi t(t-n)}\chi_{[\gamma_-,\gamma_+]}\,\mathrm{d}t + \max(0, 1-np)\delta_0 + \max(0, 1-n(1-p))\delta_n \tag{2.16}$$

where $\gamma_\pm = \left(\sqrt{(n-1)p} \pm \sqrt{1-p}\right)^2$ and χ_I denotes the characteristic function on the interval I.

Next we shall investigate the de Moivre–Laplace limiting procedure in our q-deformed binomial distribution. That is, we consider what distribution is obtained if we standardize $(\nu_{(n,p)})_q$ so as to be of mean 0 and of variance 1, and take the limit $n \to \infty$. We can perform this limiting procedure in terms of the Jacobi parameters as follows.

Consider a non-commutative random variable x in a non-commutative probability space $(\mathcal{A}, \phi)$, of which the kth moment $\phi(x^k)$ is given as one of $B_q(n, p)$, for every $k \in \mathbb{N}_0$. We will standardize x so as to have 0-expectation and to be of variance 1, that is, consider the random variable

$$z = \frac{x - \phi(x)\cdot 1}{\sqrt{\phi(x^2) - \phi(x)^2}}. \tag{2.17}$$

Find the Jacobi parameters of the orthogonal polynomials for the sequence of moments of the random variable z. Then taking the limit $n \to \infty$, we will obtain the orthogonal polynomials for the limit distribution of de Moivre–Laplace.

In our situation, if we shift so as to have 0-expectation in the q-deformed binomial distribution, its orthogonal polynomials $\{Q_m(X)\}_{m\geqslant 0}$ are given as $\{P_m(X+np)\}_{m\geqslant 0}$, of which Jacobi parameters are represented as $((\alpha_m^{(n,p)})_q - np)$ and $(\beta_m^{(n,p)})_q$. In addition, to standardize so as to be of variance 1, it can be realized by replacing the Jacobi parameters $((\alpha_m^{(n,p)})_q - np)$ and $(\beta_m^{(n,p)})_q$ by $((\alpha_m^{(n,p)})_q - np)/\sqrt{(\beta_1^{(n,p)})_q}$ and $(\beta_m^{(n,p)})_q/(\beta_1^{(n,p)})_q$, respectively, because the variance of $B_q(n, p)$ is $(\beta_1^{(n,p)})_q = np(1-p)$. Thus the orthogonal polynomials of this standardized q-deformed binomial distribution are determined by the recurrence relation that

$$\begin{aligned} &P_0(X) = 1 \qquad P_1(X) = X \\ &P_{m+1}(X) = \left(X - \frac{(1-2p)[m]_q}{\sqrt{np(1-p)}}\right)P_m(X) \\ &\qquad - \left(\frac{[m]_q\left(n - [m-1]_q\right)p(1-p)}{np(1-p)}\right)P_{m-1}(X) \qquad (m \geqslant 1). \end{aligned} \tag{2.18}$$

Taking the limit $n \to \infty$, we obtain the recurrence relation of the orthogonal polynomials for the limit distribution, namely

$$\begin{aligned} &P_0(X) = 1 \qquad P_1(X) = X \\ &P_{m+1}(X) = XP_m(X) - [m]_q P_{m-1}(X) \qquad (m \geqslant 1) \end{aligned} \tag{2.19}$$

which defines nothing but certain q-Hermite polynomials. Thus the limit distribution is a q-deformed Gaussian distribution as expected.

3. A q-deformed Poisson distribution

In this section, we define our q-deformed Poisson distribution based on the orthogonal polynomials for $B_q(n, p)$. We shall give it by taking the Poisson limits in the Jacobi parameters (2.14), that is $n \to \infty$ and $p \to 0$ but $np = \lambda > 0$ remains finite, hence

$$\left(\alpha_m^{(n,p)}\right)_q = np + (1 - 2p)[m]_q \longrightarrow \lambda + [m]_q \tag{3.1}$$

$$\left(\beta_m^{(n,p)}\right)_q = [m]_q \left(n - [m-1]_q\right) p(1-p) \longrightarrow \lambda[m]_q. \tag{3.2}$$

Definition 3.1. *For $q \in [0, 1)$ and $\lambda > 0$, we call the induced probability measure μ_q from the sequence of polynomials*

$$\begin{aligned} &P_0(X) = 1 \qquad P_1(X) = X - \lambda \\ &P_{m+1}(X) = \left(X - (\lambda + [m]_q)\right) P_m(X) - \lambda[m]_q P_{m-1}(X) \qquad (m \geqslant 1) \end{aligned} \tag{3.3}$$

the q-deformed Poisson distribution with parameter λ. In the limit $q \to 1$, the recurrence relation (3.3) defines well known classical discrete orthogonal polynomials, namely the Charlier polynomials (see, for example, [17], chapter VI-1).

Now let us find the probability measure of this q-deformed Poisson distribution. For this purpose, we will use the Al-Salam–Chihara polynomials. Al-Salam and Chihara in [19] defined the orthogonal polynomials $P_m(X) \equiv P_m(X; q; a, b, c)$ which satisfy the three-term recurrence relation

$$\begin{aligned} &P_0(X) = 1 \qquad P_1(X) = X - a \\ &P_{m+1}(X) = \left(X - aq^m\right) P_m(X) - (c - bq^{m-1})(1 - q^m) P_{m-1}(X) \qquad (m \geqslant 1) \end{aligned} \tag{3.4}$$

as the result for the characterization problem of some convolution formulae. They were unable to find the induced probability measure except when $a = b = 0$. The case $a = b = 0$ is a special case of the q-ultraspherical polynomials, namely the q-Hermite polynomials.

If we put $Q_m(X) = P_m(X + (\lambda + 1/(1-q)))$ in (3.3) then we have the relation,

$$\begin{aligned} &Q_0(X) = 1 \qquad Q_1(X) = X - \left(\frac{-1}{1-q}\right) \\ &Q_{m+1}(X) = \left(X - \left(\frac{-1}{1-q}\right) q^m\right) Q_m(X) - \frac{\lambda}{1-q}(1 - q^m) Q_{m-1}(X) \qquad (m \geqslant 1) \end{aligned} \tag{3.5}$$

which means that $\{Q_m\}_{m \geqslant 0}$ are the Al-Salam–Chihara polynomials of parameters

$$a = \frac{-1}{1-q} \qquad b = 0 \qquad c = \frac{\lambda}{1-q}. \tag{3.6}$$

Askey and Ismail in [13] succeeded in giving the distribution function for the Al-Salam–Chihara polynomials in the general case. It enables us to obtain the probability measure induced from the orthogonal polynomials $\{Q_m\}_{m \geqslant 0}$. Then shifting it $(\lambda + 1/(1-q))$ to the right, we have the probability measure for our q-deformed Poisson distribution.

At first, we consider the point masses for $\{Q_m\}_{m \geqslant 0}$. In our case, the parameters defined in equation (3.8) of [13, p 18], which are distinguished in our paper by the subscript AI, become $\alpha_{\mathrm{AI}} = -q^k/\lambda$, $\beta_{\mathrm{AI}} = (q-1)/q^k$, $\lambda_{\mathrm{AI}} = q - 1$ and $\mu_{\mathrm{AI}} \to \infty$, and it is assumed that $0 < q < 1$ and $\lambda(1-q) \leqslant 1$. Using equation (3.28), upper line, in [13, p 23] with these parameters, it follows that the discrete spectrum of the distribution function is the set of points $\{(t_k)_q\}$,

$$(t_k)_q = -\frac{q^k}{1-q} - \frac{\lambda}{q^k} \qquad k = 0, 1, \ldots, K \tag{3.7}$$

where $K = \sup\{\, k \mid q^{2k} \geqslant \lambda(1-q)\}$, and the jump $(J_k)_q$ at each point $(t_k)_q$ can be given as

$$(J_k)_q = \left(1 - \frac{\lambda(1-q)}{q^{2k}}\right) \frac{\left(\lambda(1-q)q^{-k+1}\right)^k}{(q;q)_k} \left(\lambda(1-q)q^{-k+1}; q\right)_\infty$$
$$= \left(1 - \frac{\lambda(1-q)}{q^{2k}}\right) \frac{\left(\lambda q^{-k+1}\right)^k}{[k]_q!} \frac{1}{\exp_q\left(\lambda q^{-k+1}\right)} \tag{3.8}$$

where we use the equality (2.5) for the q-deformed exponential function. In the case of $\lambda(1-q) > 1$, there is no point spectrum.

Furthermore, the results in section 3.4 in [13] derive that the absolutely continuous part of the probability measure is supported on the interval

$$\left[-2\sqrt{\frac{\lambda}{1-q}}, 2\sqrt{\frac{\lambda}{1-q}}\,\right]$$

and that letting

$$t = 2\sqrt{\frac{\lambda}{1-q}}\cos\theta \qquad (0 \leqslant \theta \leqslant \pi)$$

the density can be given as

$$\frac{1}{2\pi}\frac{(q;q)_\infty}{2\sqrt{\lambda/(1-q)}\sin\theta}\left|\frac{\left(\mathrm{e}^{-\mathrm{i}2\theta};q\right)_\infty}{\left((-1/\sqrt{\lambda(1-q)}\,)\mathrm{e}^{-\mathrm{i}\theta};q\right)_\infty}\right|^2$$
$$= \frac{1}{4\pi\sqrt{\lambda/(1-q)}\sin\theta}\frac{\left(q, \mathrm{e}^{-\mathrm{i}2\theta}, \mathrm{e}^{\mathrm{i}2\theta};q\right)_\infty}{\left((-1/\sqrt{\lambda(1-q)}\,)\mathrm{e}^{-\mathrm{i}\theta}, (-1/\sqrt{\lambda(1-q)}\,)\mathrm{e}^{\mathrm{i}\theta};q\right)_\infty}. \tag{3.9}$$

We can rewrite this density as follows:

$$\frac{1}{2\pi}\frac{(q;q)_\infty}{2\sqrt{\lambda/(1-q)}\sin\theta}\prod_{n=0}^{\infty}\left|\frac{1-\mathrm{e}^{-\mathrm{i}2\theta}q^n}{1+(1/\sqrt{\lambda(1-q)}\,)\mathrm{e}^{-\mathrm{i}\theta}q^n}\right|^2$$
$$= \frac{1-q}{2\pi}\left(\frac{2\sqrt{\lambda/(1-q)}\sin\theta}{2\sqrt{\lambda/(1-q)}\cos\theta + \lambda + 1/(1-q)}\right)$$
$$\times\prod_{n=1}^{\infty}(1-q^n)\left|\frac{1-\mathrm{e}^{-\mathrm{i}2\theta}q^n}{1+(1/\sqrt{\lambda(1-q)}\,)\mathrm{e}^{-\mathrm{i}\theta}q^n}\right|^2$$
$$= \frac{1-q}{2\pi}\left(\frac{\sqrt{4\lambda/(1-q)-t^2}}{t+\lambda+1/(1-q)}\right)\prod_{n=1}^{\infty}(1-q^n)\left(\frac{\lambda(1+q^n)^2-(1-q)q^n t^2}{q^n t+\lambda+q^{2n}/(1-q)}\right). \tag{3.10}$$

After a shift in X by $(\lambda + 1/(1-q))$ to the right in the discrete part and in the continuous part we obtain the probability measure for our q-deformed Poisson distribution.

Theorem 3.1. *For $0 < q < 1$ and $\lambda > 0$, the probability measure μ_q for the q-deformed Poisson distribution of parameter λ can be given as follows. We set the function $f_q(t)$ as*

$$f_q(t) = \frac{(1-q)\sqrt{4\lambda/(1-q)-(t-\lambda-1/(1-q))^2}}{2\pi t}$$
$$\times\prod_{n=1}^{\infty}(1-q^n)\frac{\lambda(1+q^n)^2-(1-q)q^n\,(t-\lambda-1/(1-q))^2}{q^n\,(t-\lambda-1/(1-q))+\lambda+q^{2n}/(1-q)}\chi_{I_q} \tag{3.11}$$

with the characteristic function χ_{I_q} *on the interval*

$$I_q = \left[-2\sqrt{\frac{\lambda}{1-q}} + \lambda + \frac{1}{1-q},\ 2\sqrt{\frac{\lambda}{1-q}} + \lambda + \frac{1}{1-q} \right]$$
$$= \left[\left(\sqrt{\lambda} - \sqrt{\frac{1}{1-q}} \right)^2, \left(\sqrt{\lambda} + \sqrt{\frac{1}{1-q}} \right)^2 \right] \tag{3.12}$$

and put

$$(p_k)_q = (t_k)_q + \lambda + \frac{1}{1-q} = [k]_q + \lambda \left(1 - \frac{1}{q^k} \right). \tag{3.13}$$

Then we have

$$\mu_q(\mathrm{d}t) = \begin{cases} f_q(t)\,\mathrm{d}t + \sum_{k=0}^{K} (J_k)_q \delta_{(p_k)_q} & \textit{if} \quad \lambda(1-q) \leqslant 1 \\ f_q(t)\,\mathrm{d}t & \textit{if} \quad \lambda(1-q) > 1 \end{cases} \tag{3.14}$$

where dt *denotes the Lebesgue measure and* $\delta_{(p_k)_q}$ *is the Dirac unit mass at* $t = (p_k)_q$. *Of course,*

$$K = \sup\{k \mid q^{2k} \geqslant \lambda(1-q)\} \tag{3.15}$$

and

$$(J_k)_q = \left(1 - \frac{\lambda(1-q)}{q^{2k}} \right) \frac{\left(\lambda q^{-k+1}\right)^k}{[k]_q!} \frac{1}{\exp_q\left(\lambda q^{-k+1}\right)} \tag{3.16}$$

are as we set before.

Example 3.1. It is easy to see that in the case of $q \to 1$ the absolutely continuous part vanishes and the distribution is supported on infinite discrete points k $(k = 0, 1, 2, \ldots)$ with the mass $(J_k)_1 = \mathrm{e}^{-\lambda}\lambda^k/k!$. Thus we can recover the usual Poisson distribution,

$$\mu_1(\mathrm{d}t) = \sum_{k=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^k}{k!} \delta_k. \tag{3.17}$$

On the other hand, taking the limit $q \to 0$ the density function of the absolutely continuous part becomes

$$f_0(t) = \frac{\sqrt{4\lambda - (t - \lambda - 1)^2}}{2\pi t} \chi_{\left[(\sqrt{\lambda}-1)^2, (\sqrt{\lambda}+1)^2\right]}. \tag{3.18}$$

The point mass at $(p_k)_0$ will survive for k such that $q^{2k}/(1-q) \geqslant \lambda$ holds, and as $q \to 0$ the left-hand side of the inequality tends to 0 if $k \neq 0$, and to 1 if $k = 0$. Hence we have the probability measure,

$$\mu_0(\mathrm{d}t) = f_0(t)\,\mathrm{d}t + \max(1-\lambda, 0)\delta_0 \tag{3.19}$$

for the case $q = 0$, which is nothing but the free Poisson distribution (see, for instance, section 3.7 in [2]).

4. Concluding remarks

In this paper, we have introduced a q-deformed Poisson distribution as the associated probability measure with a certain q-deformed sequence of well known classical discrete orthogonal polynomials, namely the Charlier polynomials (see definition 3.1). Several kinds of q-deformations of the Charlier polynomials have been studied by many authors. One of the most well known ones are called the classical q-deformed Charlier polynomials defined by

$$c_m(X;\lambda;q) = {}_2\phi_1(q^{-m}, X; 0; q, -q^{m+1}/\lambda) \tag{4.1}$$

(see, for instance, [15, p 187]). The Jacobi parameters of the three-term recurrence relation for the monic form $C_m^{\lambda}(X;q)$ of these polynomials can be given as

$$\begin{aligned} \alpha_m &= \lambda q^{-1-2m} + q^{-m} + \lambda q^{-2m} - \lambda q^{-m} \\ \beta_m &= \lambda q^{1-3m}(1-q^m)(1+\lambda q^{-m}). \end{aligned} \tag{4.2}$$

If we replace the variable X by $(1-q)X+1$ and the parameter λ by $(1-q)\lambda$ then the Jacobi parameters of the induced sequence of polynomials become

$$\begin{aligned} \alpha_m &= \lambda q^{-1-2m} + [m]_q q^{-m}\left(1+\lambda(1-q)q^{-m}\right) \\ \beta_m &= \lambda [m]_q q^{1-3m}\left(1+\lambda(1-q)q^{-m}\right) \end{aligned} \tag{4.3}$$

which can be regarded as a certain q-deformation of $\lambda + m$ and λm, respectively (see, for instance, [20]). These parameters, however, are still considerably different from ours (cf equation (3.3)). We can also find in [20] another q-deformation of the Charlier polynomials, which is defined by the Jacobi parameters

$$\alpha_m = \lambda q^m + [m]_q \qquad \beta_m = \lambda [m]_q q^{m-1}. \tag{4.4}$$

This deformation and ours are apparently alike, but they are radically different. Because the orthogonal polynomials defined by the Jacobi parameters (4.4) are rescaled versions of the Al-Salam–Carlitz polynomials (see [17, p 196]), whose measure is still discrete, and their q-deformation is not compatible with the free probability theory in the case of $q = 0$. Our q-deformed Poisson distribution, however, has an absolute continuous part in general, and it is closely related to the q-deformation of the Fock space. Actually, we have realized our q-deformed Poisson random variable recently in [21] as the linear combination of the q-creation, the q-annihilation, the q-number and the scalar operators on the q-Fock space.

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